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Dimension of the Global Attractor for Discretization of Damped Sine-Gordon Equation

SHENGFAN ZHOU
Department of Mathematics, Sichuan Union University
Chengdu, 610064, P.R. China

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Abstract—A more precise estimate on the dimension of the global attractor for discretization of damped sine-Gordon equation with the periodic boundary condition is obtained. The gained Hausdorff dimension remains small for large damping and is independent of the mesh sizes.

Keywords—Sine-Gordon equation, Finite difference, Global attractor, Hausdorff dimension.

1. INTRODUCTION AND THEOREM

Consider the damped sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - \Delta u + \sin u = f, \qquad x \in (0,1), \quad t \geq 0, \tag{1.1}$$

with the periodic boundary condition

$$u(0,t) = u(1,t), \qquad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t), \quad t > 0, \tag{1.2}$$

and the initial value condition

$$u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in (0,1), \tag{1.3}$$

where $u = u(x,t)$ is a real-valued function on $(0,1) \times [0,+\infty)$, $f : f(x) \in L^2(0,1)$, $\alpha > 0$, $D(-\Delta) = H^2_{\text{per}}(0,1)$, the space of H^2 functions which are spatially periodic.

The spatially finite-difference discretized version of problems (1.1) and (1.2) can be written as

$$\frac{d^2 u}{dt^2} + \alpha \frac{du}{dt} + Au + \sin u = \Gamma, \tag{1.4}$$

and the initial value condition (1.3) as

$$u(0) = u^{(0)}, \qquad \frac{du}{dt}(0) = u^{(1)}, \tag{1.5}$$

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where $u = (u_1, u_2, \ldots, u_m)^\top \in R^m$, $u^{(i)} = (u_1^{(i)}, u_2^{(i)}, \ldots, u_m^{(i)})^\top \in R^m$, $(i = 0, 1)$, $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_m)^\top \in R^m$, the sampling of function f, with $(1/m)\sum_{k=1}^m \Gamma_k^2$ uniformly bounded with respect to $m, \sin u = (\sin u_1, \sin u_2, \ldots, \sin u_m)^\top$; and

$$A = m^2 \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & -1 \\ -1 & 2 & -1 & \cdots & \cdots & \cdots & 0 \\ 0 & -1 & 2 & \ddots & . & . & \vdots \\ \vdots & . & \ddots & \ddots & \ddots & . & \vdots \\ \vdots & . & . & \ddots & 2 & -1 & 0 \\ 0 & \cdots & \cdots & \cdots & -1 & 2 & -1 \\ -1 & \cdots & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}_{m\times m}, \tag{1.6}$$

which is a semipositive, self-adjoint linear operator on R^m.

Yan in [1] proved the existence of the global attractor for systems (1.4) and (1.5) and gave an upper bound of Hausdorff dimension of the attractor for $\alpha > 0$. It reads:

$$d_H \leq 2 + \frac{32}{3\varepsilon^2} + \frac{2}{\varepsilon^2}\sum_{j=1}^{\infty}\frac{1}{j^2}, \tag{1.7}$$

where $\varepsilon = \min\{3/\alpha, \alpha/2\}$. Obviously, this upper bound in (1.7) is directly proportional to the coefficient α of damping when $\alpha \geq \sqrt{6}$, and tends to infinity as $\alpha \to +\infty$, which does not conform to the physics.

In this paper, we obtain a more strict upper bound of the dimension for the global attractor by carefully estimating and splitting the positivity of the linear operator in the corresponding evolution equation of the first order in time. The idea of using such a technique is due to Wang and Zhu [2]. The result is the following theorem.

THEOREM 1. *The Hausdorff dimension d_H of the global attractor for systems (1.4) and (1.5) satisfies*

$$d_H \leq 2 + \min\left\{\ell \mid \ell \in N,\ \frac{1}{\ell}\sum_{j=1}^{[\ell/2]+1}\frac{1}{j^2} \leq \frac{4\lambda_1\alpha^2}{\sqrt{\alpha^2+4\lambda_1}\left(\alpha+\sqrt{\alpha^2+4\lambda_1}\right)}\right\}. \tag{1.8}$$

In particular, if α satisfies

$$4\lambda_1\alpha^2 > \sqrt{\alpha^2+4\lambda_1}\left(\alpha+\sqrt{\alpha^2+4\lambda_1}\right), \tag{1.9}$$

then $d_H \leq 2$. Where $\lambda_1 = 4m^2\sin^2(\pi/m)$, $[\ell/2]$ is the largest integer which is less than or equal to $m/2$.

It is easy to see from (1.9) that $d_H \leq 2$ for sufficiently large α because

$$\frac{4\lambda_1\alpha^2}{\sqrt{\alpha^2+4\lambda_1}\left(\alpha+\sqrt{\alpha^2+4\lambda_1}\right)} \to 2\lambda_1 \geq 32,$$

as $\alpha \to +\infty$.

2. PROOF OF THE THEOREM

We have known in [1] that 0 is a simple eigenvalue of operator A, the corresponding eigenvector $e = (1, 1, \ldots, 1)^\top \in R^m$, and $\lambda_k = 4m^2\sin^2(k\pi/m)$, $k = 1, 2, \ldots, [m/2]$, are double eigenvalues of A, where $[m/2]$ is the largest integer ℓ with $\ell \leq m/2$, and if m is an even number, then $4m^2$ is a simple eigenvalue of A. Since $\sin x \geq (2/\pi)x$ for $x \in [0, \pi/2]$,

$$16 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_{[m/2]} \leq 4m^2.$$

Let $z, z^{(1)}, z^{(2)} \in R^m$ with their components $z_k, z_k^{(1)}, z_k^{(2)}, k = 1, 2, \ldots, m$. We define the inner products and norms as

$$\left(z^{(1)}, z^{(2)}\right) = \frac{1}{m} \sum_{k=1}^{m} z_k^{(1)} z_k^{(2)},$$

$$|z| = (z, z)^{1/2} = \left(\frac{1}{m} \sum_{k=1}^{m} z_k^2\right)^{1/2},$$

$$\|z\| = (Az, z)^{1/2} = \left(\frac{1}{m} \sum_{k=1}^{m} m^2 (z_{k+1} - z_k)^2\right)^{1/2},$$

in which $z_{m+1} = z_1$.

Write $\widetilde{E} = \{e\}^{\perp R^m}$, the orthogonal complement of span$\{e\}$ in R^m, which is an invariant subspace of the linear operator A. It is easy to see that $|\cdot|$ is a norm in R^m, $\|\cdot\|$ is only a seminorm in R^m, but it is a norm in $\widetilde{E}$. We also have the following inequality:

$$\|z\|^2 \geq \lambda_1 |z|^2, \qquad \forall\, z \in \widetilde{E}, \tag{2.1}$$

which corresponds to the Poincáre inequality.

Let $E_0 = (\widetilde{E}, |\cdot|)$, $E_1 = (\widetilde{E}, \|\cdot\|)$, and $V_0 = (E_1 \times T^1) \times (E_0 \times R)$, $\widetilde{V_0} = E_1 \times E_0$, where $T^1 = R^1/2\pi Z$ is the one-dimensional torus. Introduce an orthogonal projector $P : R^m \mapsto \{e\}^{\perp R^m} = \widetilde{E}$, which induces a projector from V_0 to $\widetilde{V_0}$ (also denoted by P). Write $\overline{u} = Pu$, $\overline{\Gamma} = P\Gamma$, then $\overline{u} = u - ((1/m)\sum_{j=1}^m u_j)e$, $\overline{\Gamma} = \Gamma - ((1/m)\sum_{j=1}^m \Gamma_j)e$ and the projection of system (1.4) to $\widetilde{E}$ is

$$\frac{d^2\overline{u}}{dt^2} + \alpha\frac{d\overline{u}}{dt} + A\overline{u} + \sin u - \left(\frac{1}{m}\sum_{j=1}^{m} \sin u_j\right) e = \overline{\Gamma}, \tag{2.2}$$

and the initial value condition is

$$\overline{u}(0) = \overline{u^{(0)}}, \qquad \frac{d\overline{u}}{dt}(0) = \overline{u^{(1)}}, \tag{2.3}$$

where $\overline{u^{(i)}} = u^{(i)} - ((1/m)\sum_{j=1}^m u_j^{(i)})e$, $(i = 0, 1)$.

It has been proven in [1] that the nonlinear semiflow $S(t) : (u^{(0)}, u^{(1)}) \in V_0 \to (u(t), \frac{du}{dt}(t)) \in V_0$, $t \geq 0$, determined by system (1.4) and (1.5) possesses a global attractor β in V_0, and the Hausdorff dimension $d_H(\beta)$ of $\beta \subset V_0$ satisfies

$$d_H(\beta) \leq d_H(P\beta) + 2. \tag{2.4}$$

Here $P\beta$ is exactly the global attractor of systems (2.2),(2.3) in $\widetilde{V_0}$. So, we only need to consider the Hausdorff dimension $d_H(P\beta)$.

Let $\varphi = (\overline{u}, \overline{v})^\top$, $\overline{v} = \overline{u_t} + \varepsilon\overline{u}$, where ε is chosen as

$$\varepsilon = \frac{\lambda_1 \alpha}{\alpha^2 + 4\lambda_1}, \tag{2.5}$$

then system (2.2) can be written as

$$\varphi_t + \Lambda\varphi + G(\varphi) = H, \tag{2.6}$$

where $H = (0, \overline{\Gamma})^\top$, $G(\varphi) = (0, \sin u - ((1/m)\sum_{j=1}^m \sin u_j)e)^\top$,

$$\Lambda = \begin{pmatrix} \varepsilon I & -I \\ A - \varepsilon(\alpha - \varepsilon)I & (\alpha - \varepsilon)I \end{pmatrix}. \tag{2.7}$$

By (2.1), we can define the inner product and norm in $\widetilde{V_0}$ as

$$(\varphi, \psi)_{\widetilde{V_0}} = \left(A|_{\widetilde{E}}\, \overline{u_1}, \overline{u_2}\right) + \left(\overline{v_1}, \overline{v_2}\right), \qquad |\varphi|_{\widetilde{V_0}} = (\varphi, \varphi)_{\widetilde{V_0}}^{1/2}, \tag{2.8}$$

for $\varphi = (\overline{u_1}, \overline{v_1})^\top$, $\psi = (\overline{u_2}, \overline{v_2})^\top \in \widetilde{V_0}$.

LEMMA 2. *For any* $\varphi = (\overline{u}, \overline{v})^\top \in \widetilde{V_0}$,

$$(\Lambda\varphi, \varphi)_{\widetilde{V_0}} \geq \sigma|\varphi|^2_{\widetilde{V_0}} + \frac{\alpha}{2}|\overline{v}|^2, \tag{2.9}$$

where

$$\sigma = \frac{\lambda_1\alpha}{\sqrt{\alpha^2+4\lambda_1}\left(\alpha+\sqrt{\alpha^2+4\lambda_1}\right)}. \tag{2.10}$$

PROOF. From (2.7) and (2.8), we have

$$\begin{aligned}(\Lambda\varphi, \varphi)_{\widetilde{V_0}} - \sigma|\varphi|^2_{\widetilde{V_0}} - \frac{\alpha}{2}|\overline{v}|^2 &= (\varepsilon-\sigma)\|\overline{u}\|^2 + \left(\frac{\alpha}{2}-\varepsilon-\sigma\right)|\overline{v}|^2 - \varepsilon(\alpha-\varepsilon)(\overline{u},\overline{v}) \\ &\geq \text{ by (2.1)} \\ &\geq (\varepsilon-\sigma)\|\overline{u}\|^2 + \left(\frac{\alpha}{2}-\varepsilon-\sigma\right)|\overline{v}|^2 - \frac{\varepsilon(\alpha-\varepsilon)}{\sqrt{\lambda_1}}\|\overline{u}\|\cdot|\overline{v}| \\ &\geq (\varepsilon-\sigma)\|\overline{u}\|^2 + \left(\frac{\alpha}{2}-\varepsilon-\sigma\right)|\overline{v}|^2 - \frac{\varepsilon\alpha}{\sqrt{\lambda_1}}\|\overline{u}\|\cdot|\overline{v}|.\end{aligned}$$

A simple computation by (2.5) and (2.10) shows

$$4(\varepsilon-\sigma)\left(\frac{\alpha}{2}-\varepsilon-\sigma\right) = \frac{\varepsilon^2\alpha^2}{\lambda_1}.$$

Thus, the proof is completed.

LEMMA 3. *For any orthonormal family of elements of* $\widetilde{V_0}$, $\{\xi_j, \eta_j\}_{j=1}^{\ell}$,

$$\sum_{j=1}^{\ell}|\xi_j|^2 \leq \sum_{j=1}^{\ell}\frac{1}{\widetilde{\lambda}_j}, \tag{2.11}$$

where $0 < \widetilde{\lambda}_1 \leq \widetilde{\lambda}_2 \leq \cdots \leq \widetilde{\lambda}_\ell \leq \cdots \leq \widetilde{\lambda}_{m-1}$ *are eigenvalues of operator* $A|_{\widetilde{E}}$.

PROOF. See [3, Lemma VI.6.3].

To estimate the dimension of the attractor for system (2.2), we consider the first variation equation of (2.2)

$$\Psi' = \overline{F'(\varphi)(U,V)^\top} = -\Lambda\Psi + G'(\varphi)(U,V)^\top, \qquad \Psi(0) = \left(\overline{\xi},\overline{\eta}\right)^\top \in \widetilde{V_0}, \tag{2.12}$$

where $\Psi = (\overline{U}, \overline{V})^\top$, $\varphi = (\overline{u}, \overline{v})^\top$ is a solution of (2.2),(2.3),

$$G'(\varphi)(U,V)^\top = \left(0, \begin{pmatrix}(\cos u_1)U_1 \\ (\cos u_2)U_2 \\ \vdots \\ (\cos u_m)U_m\end{pmatrix} - \left(\frac{1}{m}\sum_{j=1}^{m}(\cos u_j)\,U_j\right)e\right)^\top, \tag{2.13}$$

$u = (u_1, u_2, \ldots, u_m)^\top$ is a solution of (1.4),(1.5) $U = (U_1, U_2, \ldots, U_m)^\top$, $V = U' + \varepsilon U$ is a solution of the variation equation of (1.4),(1.5) with initial value conditions $U(0) = \xi = (\xi_1, \xi_2, \ldots, \xi_m)^\top$, $V(0) = U'(0) + \varepsilon U(0) = \eta = (\eta_1, \eta_2, \ldots, \eta_m)^\top$,

$$\overline{U} = U - \frac{1}{m}\sum_{j=1}^{m}U_j, \qquad \overline{\xi} = \xi - \frac{1}{m}\sum_{j=1}^{m}\xi_j, \qquad \overline{\eta} = \eta - \frac{1}{m}\sum_{j=1}^{m}\eta_j.$$

LEMMA 4. *The Hausdorff dimension $d_H(P\beta)$ of the global attractor for system (2.2),(2.3) satisfies*

$$d_H(P\beta) \le \min\left\{ \ell \mid \ell \in N,\ \frac{1}{\ell}\sum_{j=1}^{[\ell/2]+1}\frac{1}{j^2} \le \frac{4\lambda_1\alpha^2}{\sqrt{\alpha^2+4\lambda_1}\left(\alpha+\sqrt{\alpha^2+4\lambda_1}\right)} \right\}, \tag{2.14}$$

where $\lambda_1 = 4m^2\sin^2(\pi/m)$.

PROOF. Let $\ell \in N$ be fixed. Consider ℓ solutions $\Psi_1, \Psi_2, \dots, \Psi_\ell$ of (2.12). At a given time τ, let $Q_\ell(\tau)$ be the orthogonal projector in $\widetilde{V_0}$ onto the space spanned by $\Psi_1, \Psi_2, \dots, \Psi_\ell$. Let $\Phi^j(\tau) = (\overline{\xi^j}(\tau), \overline{\eta^j}(\tau))^\top \in \widetilde{V_0}$, $j = 1, 2, \dots, \ell$, denote an orthonormal basis of $Q_\ell(\tau)\widetilde{V_0} = \operatorname{span}\{\Psi_1(\tau), \Psi_2(\tau), \dots, \Psi_\ell(\tau)\}$. Consider

$$\begin{aligned} \operatorname{Tr}\overline{F'(\varphi(\tau))}\circ Q_\ell(\tau) &= \sum_{j=1}^{\ell}\left(\overline{F'(\varphi(\tau))\,\Phi^j(\tau)}, \Phi^j(\tau)\right)_{\widetilde{V_0}} \\ &= -\sum_{j=1}^{\ell}\left[\left(\Lambda\Phi^j, \Phi^j\right)_{\widetilde{V_0}} - \left(G'(\varphi)\,\Phi^j, \Phi^j\right)_{\widetilde{V_0}}\right]. \end{aligned}$$

By Lemma 2 and $|\Phi^j|_{\widetilde{V_0}} = 1$,

$$\left(\Lambda\Phi^j, \Phi^j\right)_{\widetilde{V_0}} \le \sigma + \frac{\alpha}{2}\left|\overline{\eta^j}\right|^2.$$

By (2.8) and (2.13),

$$\begin{aligned} \left|\left(G'(\varphi)\Phi^j, \Phi^j\right)_{\widetilde{V_0}}\right| &= \left|\left(\begin{pmatrix} (\cos u_1)\overline{\xi_1^j} \\ (\cos u_2)\overline{\xi_2^j} \\ \vdots \\ (\cos u_m)\overline{\xi_m^j} \end{pmatrix} - \left(\frac{1}{m}\sum_{i=1}^{m}(\cos u_i)\overline{\xi_i^j}\right)e,\ \overline{\eta^j}\right)\right| \\ &\le \left|\frac{1}{m}\sum_{i=1}^{m}(\cos u_i)\overline{\xi_i^j}\cdot\overline{\eta_i^j}\right| + \left|\left(\frac{1}{m}\sum_{i=1}^{m}(\cos u_i)\,\overline{\xi_i^j}\right)\left(\frac{1}{m}\sum_{i=1}^{m}\overline{\eta_i^j}\right)\right| \\ &\le \left|\overline{\xi^j}\right|\cdot\left|\overline{\eta^j}\right| + |\cos u|\cdot\left|\overline{\xi^j}\right|\cdot|e|\cdot\left|\overline{\eta^j}\right| \\ &\le 2\left|\overline{\xi^j}\right|\cdot\left|\overline{\eta^j}\right|. \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{Tr}\overline{F'(\varphi(\tau))}\circ Q_\ell(\tau) &\le -\ell\sigma - \frac{\alpha}{2}\sum_{j=1}^{\ell}\left|\overline{\eta^j}\right|^2 + \sum_{j=1}^{\ell}2\left|\overline{\xi^j}\right|\cdot\left|\overline{\eta^j}\right| \\ &\le -\ell\sigma + \frac{2}{\alpha}\sum_{j=1}^{\ell}\left|\overline{\xi^j}\right|^2 \\ &\le \text{(by (2.11))} \\ &\le -\ell\sigma + \frac{2}{\alpha}\sum_{j=1}^{\ell}\frac{1}{\widetilde{\lambda_j}}. \end{aligned}$$

Since $\widetilde{\lambda_1} = \widetilde{\lambda_2} = \lambda_1$, $\widetilde{\lambda_3} = \widetilde{\lambda_4} = \lambda_2, \dots$, and $\lambda_k = 4m^2\sin^2(k\pi/m) \ge 16k^2$, $k = 1, 2, \dots, [m/2]$,

$$\operatorname{Tr}\overline{F'(\varphi(\tau))}\circ Q_\ell(\tau) \le -\ell\sigma + \frac{1}{4\alpha}\sum_{j=1}^{[\ell/2]+1}\frac{1}{\lambda_j}. \tag{2.15}$$

If $(1/\ell)\sum_{j=1}^{[\ell/2]+1}(1/j^2) \le 4\alpha\sigma = (4\lambda_1\alpha^2)/(\sqrt{\alpha^2+4\lambda_1}(\alpha+\sqrt{\alpha^2+4\lambda_1}))$, then by (2.15),

$$\mathrm{Tr}\overline{F'(\varphi(\tau))} \circ Q_\ell(\tau) \le 0, \tag{2.16}$$

i.e., the Lyapunov exponents $\mu_j,\ j \in N$ of $P\beta$ satisfies

$$\mu_1 + \mu_2 + \cdots + \mu_\ell \le 0. \tag{2.17}$$

By [3, Theorem V.3.3], (2.14) is true.

COROLLARY 5.

$$d_H(P\beta) \le \frac{\sqrt{\alpha^2+4\lambda_1}\left(\alpha+\sqrt{\alpha^2+4\lambda_1}\right)}{2\lambda_1\alpha^2}. \tag{2.18}$$

PROOF. Since $\sum_{j=1}^{[\ell/2]+1}(1/j^2) < \sum_{j=1}^{\infty}(1/j^2) < 2$, by (2.15),

$$\mathrm{Tr}\overline{F'(\varphi(\tau))} \circ Q_\ell(\tau) < -\ell\sigma + \frac{1}{2\alpha} < 0,$$

for $\ell \ge (1/2\alpha\sigma) = (\sqrt{\alpha^2+4\lambda_1}(\alpha+\sqrt{\alpha^2+4\lambda_1}))/(2\lambda_1\alpha^2)$.

COROLLARY 6. *If* $4\lambda_1\alpha^2 > \sqrt{\alpha^2+4\lambda_1}(\alpha+\sqrt{\alpha^2+4\lambda_1})$, *then* $d_H(P\beta) = 0$.

PROOF. In this case, $\ell = 1$ in (2.15) and $(\overline{F'(\varphi(\tau))\Phi(\tau)}, \Phi(\tau))_{\widetilde{V_0}} < 0$ for any unit element $\Phi = (\overline{\xi}, \overline{\eta})^{\mathsf{T}} \in \widetilde{V_0}$. So, by (2.17), $\mu_1 < 0$, hence, $d_H(P\beta) = 0$.

Combining with Lemma 4, Corollary 6, and (2.14), we complete the proof of Theorem 1.

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